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BRST symmetries for the tangent gauge group

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Abstract. For any principal bundle P, one can consider the subspace of the space of connections on its tangent bundle $\mathrm{T}P$ given by the tangent bundle $\mathrm{T}\mathcal{A}$ of the space of connections $\mathcal{A}$ on P. The tangent gauge group acts freely on $\mathrm{T}\mathcal{A}$. Appropriate BRST operators are introduced for quantum field theories that include as fields elements of $\mathrm{T}\mathcal{A}$, as well as tangent vectors to the space of curvatures. As the simplest application, the BRST symmetry of the so-called BF-Yang–Mills theory is described and the relevant gauge fixing conditions are analyzed. A brief account on the topological BF theories is also included and the relevant Batalin–Vilkovisky operator is described.

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I. Introduction

In this paper we study the BRST complexes arising from the action of the (iterated) tangent bundle of the group of gauge transformations $\mathcal{G}$ on the (iterated) tangent bundle of the space of connections $\mathcal{A}$.

Our main original motivation for such a study was the understanding of the BRST and the Batalin–Vilkovisky (BV) structure of 4-dimensional BF theories. These topological theories have been shown [1] to be strictly related to the 4-dimensional Yang–Mills theory.

Let us first recall what BF action functionals are in both three and four dimensions. We consider "space–time" to be given by a closed, oriented Riemannian manifold M with a gauge structure given by a G-principal bundle P with base M. The group G is a simple compact connected Lie Group (typically $SU(N)$) with Lie algebra denoted by $\mathfrak{g}$. Unless otherwise stated, the forms we are considering on M will take values in the adjoint bundle and hence their local expression will take values in $\mathfrak{g}$.

When $M \equiv M^3$ is a *3-dimensional* space, then the most known topological theory is that defined in terms of the Chern–Simons action functional

$$CS(A) \equiv \int_{M^3} \mathrm{Tr}\, \left(A \wedge dA + \frac{1}{3} A \wedge [A, A]\right),$$

where A is (the local expression of) a connection on $P(M, G)$.

We can also introduce a field $\widehat{B}$ given by a 1-form. The 1-form $A + t\widehat{B}$ is again a connection.

The 3-dimensional topological BF action functional is then defined in terms of the Chern–Simons action functional by

$$S_{BF,t}(A, \widehat{B}) \equiv \frac{1}{2t}\left(CS(A + t\widehat{B}) - CS(A - t\widehat{B})\right) =$$
$$\int_{M^3} \mathrm{Tr}\, \left(2(\widehat{B} \wedge F_A) + \frac{t^2}{3} \hat{B} \wedge [\hat{B}, \hat{B}]\right), \tag{I.1}$$

where F_A denotes the curvature of A and t is a real parameter. The action functional (I.1) is called the *BF action functional with cosmological constant* [2].

In the limit $t \to 0$ we obtain, up to a factor 2, the so called *pure BF action functional*

$$S_{BF} = \int_{M^3} \mathrm{Tr}\, \left(\hat{B} \wedge F_A\right).$$

One can define observables related to links in M^3 for both $S_{BF,t}$ and S_{BF}; it has been shown in [2] that the corresponding expectation values give the coefficients of the Homfly–Jones polynomials in the first case and are related to the Alexander–Conway polynomials in the second case.

In *4 dimensions* we first consider the topological Chern action functional

$$\int_M \mathrm{Tr}\, \left(F_A \wedge F_A\right).$$

Then we consider a field B given by a 2-form. The 4-dimensional topological BF *action functional (with cosmological constant)* is then defined as

$$S_{BF,t} \equiv \frac{1}{2t}\int_M \mathrm{Tr}\left((F_A + tB)\wedge(F_A + tB) - F_A\wedge F_A\right) =$$

$$\int_M \mathrm{Tr}\left(B\wedge F_A + \frac{t}{2}B\wedge B\right), \tag{I.2}$$

and again the *pure BF action functional* S_{BF} is defined as the limit of the above one when t goes to 0.

In 4-dimensions one may also start with the Yang–Mills action functional

$$S_{YM} = \int_M \mathrm{Tr}\left(F_A\wedge *F_A\right),$$

where $*$ is the Hodge operator. (We omit the coupling constant here.)

The corresponding BF-Yang–Mills ($BFYM$) action functional reads

$$S_{BFYM,t} \equiv \int_M \mathrm{Tr}\left(B\wedge F_A + \frac{t}{2}B\wedge *B\right), \tag{I.3}$$

whose limit $t\longrightarrow 0$ gives again the pure BF action functional.

When $t = 1$, (I.3) is *equivalent, by Gaussian integration, to the Yang–Mills action functional.*

When $t = 0$, (I.3) has an extra-symmetry, namely, the translation $B \longrightarrow B + d_A\tau$ where τ is a 1-form. In order to extend this symmetry for a generic t and carry on the perturbative expansion around the critical solutions for the BF-Yang–Mills theory, it is necessary to deform the action functional (I.3) into the following action functional:

$$S_{BFYM,\eta,t} \equiv \int_M \mathrm{Tr}\left(B\wedge F_A + \frac{t}{2}(B - d_A\eta)\wedge *(B - d_A\eta)\right). \tag{I.4}$$

Here η is a 1-form and the new symmetry reads, for a generic 1-form τ

$$B\longrightarrow B + d_A\tau, \quad \eta\longrightarrow \eta + \tau. \tag{I.5}$$

It is now possible to show that in perturbation theory, the action functional (I.4) is equivalent to the Yang–Mills action functional and we refer to [1] for such a discussion.

By now it is clear, just by inspecting all the action functionals written above, that in both the 3- and the 4-dimensional BF theories, the field $\hat{B}$ and respectively B belong to some tangent space to the space of fields: the connections in three dimensions and the curvatures in four.

Some care is to be applied in four dimensions, for the space of “curvatures” is not the space of fields where one performs the functional integration.

Specifically, in the $BFYM$ theories with action functional (I.4), one works with triples (A, η, B) where the pair (A, η) belongs to the tangent bundle of the space of connections and B is a tangent vector at F_A to the vector space of 2-forms.

The relevant BRST cohomology should accomodate both the gauge invariance and the invariance under translations (I.5); the latter condition implies that eventually no physical quantity should depend on the field η; that is, the theory should behave like a topological field theory as far as the field η is concerned.

The existence of such a BRST cohomology is a priory not obvious; the purpose of this paper is to describe in details such a BRST cohomology, which is nothing but a straightforward extension of the BRST cohomology arising from the action of the tangent gauge group on the tangent bundle of connections.

The plan of the paper is as follows:

In section **II** we recall some aspects of the topological field theories of cohomological type as discussed in [3].

In sections **III** and **IV** we study the space of connections of the tangent bundle of a principal bundle (tangent connections) and the relevant gauge group.

In section **V** and **VI** we discuss the BRST cohomology arising from the action of the tangent gauge group on the tangent bundle of connections.

In section **VII** we show that it is possible to extend the above BRST operator to the 2-form B.

In section **VIII** we show that the construction can be indefinitely iterated by including 2^n 2-forms B_i, 2^n 1-forms η_i and the relevant ghosts.

In section **IX** we consider the BV construction corresponding to the action functional (I.2).

In section **X** we discuss the orbit space and the moduli space relevant to the action functional (I.4).

Some of the results of sections **VI**, **VII**, **X** have been anticipated in [1].

As a final remark, we notice that the 2-form B in 4-dimensional BF theories can be interpreted geometrically as a tangent vector to the space of connections over the following bundle.

Take the space $\mathcal{P}_A(P)$ of (free) horizontal paths on P with respect to a connection A. This is a principal G-bundle, whose base space is the space of the free paths on M. The initial-point map $ev_0 : \mathcal{P}_A(P) \to P$ is a bundle morphism, so $ev_0^* A$ is a connection on $\mathcal{P}_A(P)$. Any form $B \in \Omega^2(M, \mathrm{ad}\, P)$, once integrated over the paths, represents a tangent vector to the space of connections on $\mathcal{P}_A(P)$. The computation of the relevant holonomies allows us to define observables corresponding to imbedded tori in a (simply connected) 4-manifold M, for which expectation values with respect to the different BF action functionals can be computed. We refer to a separate paper [4] for such a discussion.

II. Symmetries in gauge and topological field theories

In this section we review some well known facts concerning gauge invariance in topological (cohomological) theories and in Yang–Mills theories and the BRST structure of such theories.

The main symmetry group we are considering is the gauge group $\mathcal{G}$ that acts on the space $\mathcal{A}$ of (irreducible) smooth connections on P. We assume that we have divided $\mathcal{G}$ by its center, so that $\mathcal{G}$ acts freely on $\mathcal{A}$ yielding the principal bundle of gauge-orbits.

As usual, we denote by $\operatorname{ad} P$ the associated bundle $P \times_{Ad} \mathfrak{g}$ and by $\Omega^*(M, \operatorname{ad} P)$ the graded Lie algebra of tensorial form on P of the adjoint type, or equivalently, of forms on M with values in $\operatorname{ad} P$. For any $A \in \mathcal{A}$ the covariant exterior derivative will be a linear map

$$d_A \colon \Omega^*(M, \operatorname{ad} P) \to \Omega^{*+1}(M, \operatorname{ad} P)\,.$$

The space $\mathcal{A}$ is an affine space and its tangent space $\mathrm{T}_A\mathcal{A}$ is the vector space $\Omega^1(M, \operatorname{ad} P)$.

A quantity (observable, action functional) is gauge invariant if it is invariant under the action of the group $\mathcal{G}$. It is moreover independent of the connection if it is invariant under the translation group $\Omega^1(M, \operatorname{ad} P)$. In quantum field theories one may wish to consider simultaneously both kinds of invariance.

In this case one is naturally led to consider the semidirect product

$$\mathcal{G}_{\mathrm{T}} \triangleq \mathcal{G} \ltimes \Omega^1(M, \operatorname{ad} P)\,. \tag{II.1}$$

In this group the product of two elements (g_1, ρ_1) and (g_2, ρ_2) is defined by

$$(g_1, \rho_1)(g_2, \rho_2) = \left(g_1 g_2, \operatorname{Ad}_{g_2^{-1}} \rho_1 + \rho_2\right).$$

Here $g_i \in \mathcal{G}$ and $\rho_i \in \Omega^1(M, \operatorname{ad} P)$.

The group $\mathcal{G}_{\mathrm{T}}$ acts on the space of connections as follows:

$$A \cdot (g, \rho) = A^g + \rho \qquad \text{(II.2)},$$

where A^g denotes the gauge-transformed connection.

This action is not free since $(g, A - A^g) \in \mathcal{G}_{\mathrm{T}}$ leaves the connection A fixed. At the infinitesimal level, we see that the freedom of the action is missing since we have non-trivial solutions of the equation

$$\rho + d_A \chi = 0, \quad \rho \in \Omega^1(M, \operatorname{ad} P), \chi \in \Omega^0(M, \operatorname{ad} P),$$

where d_A is the covariant exterior derivative.

From a field-theoretical point of view, dealing with a symmetry that does not correspond to a free action is troublesome since a gauge fixing mechanism and a consistent definition of ghost fields may not be available.

The cure for this problem is to require ρ to belong to a complementary space of the image of $d_A\colon \Omega^0(M, \mathrm{ad}\, P) \to \Omega^1(M, \mathrm{ad}\, P)$. In other words we have to choose a *connection on the bundle of gauge orbits*

$$\mathcal{A} \mapsto \frac{\mathcal{A}}{\mathcal{G}}. \tag{II.3}$$

Choosing such a connection is also called *fixing the gauge.* We denote the above connection on the bundle of gauge orbits by the symbol c.

In turn any connection c on $\mathcal{A}$, determines a connection on the bundle

$$\frac{P \times \mathcal{A}}{\mathcal{G}} \mapsto M \times \frac{\mathcal{A}}{\mathcal{G}}. \tag{II.4}$$

In fact the 1-form on $P \times \mathcal{A}$ given by

$$(A + c)_{(p,A)}(X, \eta) \triangleq A(X)_p + c(\eta)_{(A,p)}, \quad X \in \mathrm{T}_p P,\ \eta \in \mathrm{T}_A \mathcal{A} \tag{II.5}$$

defines a $\mathcal{G}$-invariant connection on $P \times \mathcal{A}$ [5] , i.e., a connection on the principal G-bundle (II.4) .

The exterior derivative on forms on $P \times \mathcal{A}$ which are of order (k, s) (i.e., of order k as forms on P and order s as forms on $\mathcal{A}$) can be written as $d_{tot} = d + (-1)^k \delta$ where d and δ are the exterior derivatives on P and $\mathcal{A}$ respectively. Analogously we set the following convention for the commutator

$$[\omega_1, \omega_2] = (-1)^{k_1 k_2 + s_1 s_2 + 1} [\omega_2, \omega_1],$$

where (k_i, s_i) denotes the order of the form ω_i. As a consequence we have the following expression for the total exterior covariant derivative

$$d_{A+c} = d_A + (-1)^k \delta_c, \tag{II.6}$$

where k denotes again the order of the form on P. In particular we have to remember equation (II.6) when computing curvatures and their Bianchi identity.

The order of a form with respect to the space $\mathcal{A}$ is called the *ghost number.*

The operator δ (or some of its restrictions or extensions) is as an example of what in quantum field theory is called a BRST (or a BRS) operator; it describes the symmetry of the fields that one considers in the theory.

We have in mind essentially two cases.

The first case is that of the *topological field theory* à la Baulieau–Singer [3] or, according to Witten's terminology [6] , of a *cohomological field theory.* In this case the theory has no degrees of freedom and the action functional is invariant under (infinitesimal) translations of the space of fields. The corresponding BRST operator is then the exterior derivative on the space of fields ($\mathcal{A}$ in the previous case).

The second case is that of a field theory (generally not topological) in which there is a symmetry group acting on the space of fields and the theory is only required to be invariant under such a symmetry group. In this last case the BRST operator is just the

restriction of the exterior derivative on the space of fields to the orbits of that symmetry group.

In the Yang–Mills theory, the BRST operator is the exterior derivative along the gauge orbits.

Mixed cases will also (and especially) be considered in this paper, namely, theories for which the BRST operator is the exterior derivative for some of the fields and the derivative along the fibers (orbits) for other fields.

We may say that these theories are "semi-topological" or topological in some fields and non-topological in others. We use this terminology, even though topological field theories are not only those of cohomological type.

In a topological theory (of the cohomological type) defined for an even-dimensional space, the typical action functional is defined in terms of characteristic classes, so is independent of the choice of the connection.

Let us come back to the Baulieu–Singer theory. Such a theory is invariant under translations in the space of connections, the action functional is represented by characteristic classes, and the field equations are obtained by considering the structure equation for the connection (II.5) and the relevant Bianchi identities.

The structure equations read

$$F_A + \psi + \phi = d_A A + d_A c + \frac{1}{2}[c, c] - \delta A + \delta c, \tag{II.7}$$

where the l.h.s. gives the various components of the curvature of the connection (II.5); F_A, ψ, ϕ are respectively forms of degree $(2, 0)$, $(1, 1)$ and $(0, 2)$ on the product space

$$P \times \mathcal{A}.$$

The previous equation can be rewritten as

$$\delta A = d_A c - \psi, \quad \delta c = -\frac{1}{2}[c, c] + \phi. \tag{II.8}$$

Notice that the form ψ is minus the projection of $\Omega^1(M, \operatorname{ad} P)$ onto the horizontal subspace at A.

The Bianchi identity defines the transformation properties of ψ and ϕ (and F_A)

$$\delta\psi = d_A\phi - [\psi, c], \quad \delta\phi = [\phi, c] \tag{II.9}$$

$$d_A F_A = 0, \quad \delta F_A = [F_A, c] - d_A\psi. \tag{II.10}$$

By comparing the first equation in (II.8) with the action (II.2), we see that we have successfully turned around the problem of the non-freedom of the action (II.2).

As was recalled before, the *Yang–Mills* theory, as opposed to the previous topological theory, involves the restriction of the transformation (II.8) to the orbit passing through $A \in \mathcal{A}$. The relevant equations are then

$$\delta A = d_A c, \quad \delta c = -\frac{1}{2}[c, c], \tag{II.11}$$

where c in (II.11) is now the *Maurer–Cartan form* on $\mathcal{G}$, and δ is the exterior derivative on $\mathcal{G}$ or, more precisely, on the $\mathcal{G}$-orbit passing through A.

The equations (II.11) are the classical BRST equations [7] .

A final remark is in order. Whenever we have a non-free action of a group over a manifold, one is naturally led to consider the corresponding equivariant cohomology. In our case the group is $\mathcal{G}_{\mathrm{T}}$ with its non-free action on $\mathcal{A}$. But $\mathcal{G}_{\mathrm{T}}$ is the semidirect product of an abelian (contractible) subgroup $\Omega^1(M, \mathrm{ad}\, P)$ times $\mathcal{G}$, which behaves in many ways like a "compact group" and acts freely on (the contractible space) $\mathcal{A}$.

So one has just to consider this last action, or the cohomology of the principal bundle (II.4) (which in turn is just the equivariant cohomology corresponding to the non-free action of $\mathcal{G}$ on P).

In the following sections we will extend this procedure by considering the action of the tangent bundle of the gauge group (which also is a Lie group) on the tangent space to the space of connections. This may look irrelevant from a purely topological point of view since, as will be recalled below, the tangent gauge group is again the semidirect product of $\mathcal{G}$ times an abelian contractible group. This implies that the classifying spaces for the gauge group and for its tangent bundle are homotopically equivalent.

But the BRST structure of the tangent gauge group will show an interesting degree of flexibility, due exactly to the fact that there is a large abelian normal subgroup of the tangent gauge group. This will allow us to include into the BRST algebra the 2-form fields B of the 4-dimensional BF theories.

However, before doing so, we need to discuss some basic facts of the differential geometry of the tangent bundle of a principal bundle.

III. Tangent principal bundles and their gauge groups

Let G be a Lie group and let $\mathfrak{g}$ its Lie algebra. The tangent bundle $\mathrm{T}G$ is a Lie group with respect to the multiplication given by the tangent map of the multiplication $m\colon G \times G \mapsto G$

$$\mathrm{T}m\colon \mathrm{T}G \times \mathrm{T}G \mapsto \mathrm{T}G$$

Explicitly this multiplication is given by

$$(k, u_k)(h, v_h) \equiv (kh, kv_h + u_k h) \quad k, h \in G; u_k \in \mathrm{T}_k G; v_h \in \mathrm{T}_h G.$$

Here the right and left multiplications of a vector by an element of G denote the push-forward of the corresponding multiplications in G.

Since we can represent any vector u_k as $u_k = kx$ for a unique $x \in \mathfrak{g} = \mathrm{T}_e G$, there is an isomorphism

$$\mathrm{T}G \xrightarrow{\sim} G \ltimes \mathfrak{g}, \quad (k, u_k) \rightsquigarrow (k, k^{-1}u_k) \qquad \text{(III.1)},$$

and on the semidirect product $G \ltimes \mathfrak{g}$ the product of two elements is defined as

$$(k,x)(h,y) \equiv (kh, \mathrm{Ad}_{h^{-1}}(x)+y) \quad k,h \in G;\ x,y \in \mathfrak{g}.$$

The inverse of the element $(k,x) \in (G \ltimes \mathfrak{g})$ is given by

$$(k,x)^{-1} = (k^{-1}, -\mathrm{Ad}_k(x)), \tag{III.2}$$

and the conjugation on $G \ltimes \mathfrak{g}$ is given by

$$(k,x)(h,y)(k,x)^{-1} = \left(khk^{-1}, \mathrm{Ad}_k\mathrm{Ad}_{h^{-1}}(x) + \mathrm{Ad}_k(y) - \mathrm{Ad}_k(x)\right). \tag{III.3}$$

The adjoint action of $G \ltimes \mathfrak{g}$ on $\mathrm{T}_{(e,0)}(G \ltimes \mathfrak{g}) = \mathfrak{g} \times \mathfrak{g}$ is given by

$$\mathrm{Ad}_{(k,z)}(x,y) = (\mathrm{Ad}_k(x), \mathrm{Ad}_k([z,x]+y)), \tag{III.4}$$

whereas the commutator is given by

$$[(x_1,y_1),(x_2,y_2)] = ([x_1,y_2],[x_1,y_2]+[y_1,x_2]). \tag{III.5}$$

So $Lie(\mathrm{T}G)$ is isomorphic as a Lie algebra to the semidirect sum $\mathfrak{g} \oplus_s \mathfrak{g}$. We now consider a G-principal bundle $P \mapsto M$, with base space M. Its tangent bundle is a $\mathrm{T}G$-principal bundle with right action of $\mathrm{T}G$ on $\mathrm{T}P$ given by the tangent map to the right action $R\colon P \times G \mapsto P$,

$$\begin{aligned} \mathrm{T}R\colon \mathrm{T}P \times \mathrm{T}G &\mapsto \mathrm{T}P \\ ((p,X),(k,u_k)) &\rightsquigarrow (pk, Xk + pu_k). \end{aligned} \tag{III.6}$$

Here and in what follows, Xk and pu_k (with $p \in P$, $X \in \mathrm{T}_pP$, $k \in G$, $u_k \in \mathrm{T}_kG$) are simplified notations for the partial tangent maps $\mathrm{T}_1R\colon \mathrm{T}P \times G \to \mathrm{T}P$ and $\mathrm{T}_2R\colon P \times \mathrm{T}G \to \mathrm{T}P$. We now rewrite the previous action, by considering the isomorphism (III.1). If $(k,x) \in G \ltimes \mathfrak{g}$, and $i(x)$ denotes the fundamental vector field in P generated by $x \in \mathfrak{g}$, then the action (III.6) becomes

$$(p,X)(k,x) = \left(pk, Xk + i(x)|_{pk}\right). \tag{III.7}$$

It is easy to check directly that (III.7) defines a good action

$$[(p,X)(k,x)]\,(h,y) = \left(pkh, (Xk)h + (\,i(x)|_{pk})h + i(y)|_{pkh}\right)$$

$$= \left(pkh, X(kh) + i\,(\mathrm{Ad}_{h^{-1}}x)|_{pkh} + i(y)|_{pkh}\right) = (p,X)\,[(k,x)(h,y)].$$

The action (III.7) is free and moreover $\pi\colon \mathrm{T}P \mapsto \mathrm{T}M$ is a $(G \ltimes \mathfrak{g})$-principal fiber bundle.

We now shift to the infinite-dimensional case. We consider the tangent bundle $\mathrm{T}\mathcal{G}$ of the group of gauge transformations.

This can be identified with the semidirect product of $\mathcal{G} \ltimes \Omega^0(M, \operatorname{ad} P)$, where the product of two elements is defined by

$$(g_1, \zeta_1)(g_2, \zeta_2) = \left(g_1 g_2, \operatorname{Ad}_{g_2^{-1}}\zeta_1 + \zeta_2\right). \tag{III.8}$$

Its Lie algebra is given by the semidirect sum $\Omega^0(M, \operatorname{ad} P) \oplus_s \Omega^0(M, \operatorname{ad} P)$ whose commutator is defined as follows

$$[(\chi_1, \zeta_1), (\chi_2, \zeta_2)] = ([\chi_1, \chi_2], [\chi_1, \zeta_2] + [\zeta_1, \chi_2]).$$

The tangent map to the action of $\mathcal{G}$ on $\mathcal{A}$ defines the action of $\mathrm{T}\mathcal{G}$ on $\mathrm{T}\mathcal{A} \sim \mathcal{A} \times \Omega^1(M, \operatorname{ad} P)$ as follows

$$(A, \eta) \cdot (g, \zeta) = (A^g, \operatorname{Ad}_{g^{-1}}(\eta) + d_{A^g}\zeta). \tag{III.9}$$

The fundamental vector field at $(A, \eta) \in \mathrm{T}\mathcal{A}$ corresponding to $(\chi, \zeta) \in \mathit{Lie}(\mathrm{T}\mathcal{G})$ is given by

$$(d_A\chi, [\eta, \chi] + d_A\zeta). \tag{III.10}$$

The action (III.9) is free when we understand that only irreducible connections on P are taken into account and that the group $\mathcal{G}$ has been divided by its center.

A *gauge transformation* for $\mathrm{T}P$ is, by definition, a map

$$f\colon \mathrm{T}P \longrightarrow G \ltimes \mathfrak{g} \quad \text{such that} \quad f[(p, X)(k, x)] = (k, x)^{-1} f(p, X)(k, x).$$

We have the following

Theorem III.1. *The tangent bundle $\mathrm{T}\mathcal{G}$ of the gauge Lie group $\mathcal{G}$ is a proper subgroup of the group $\mathcal{G}_{\mathrm{T}P}$ of gauge transformations for $\mathrm{T}P$.*

Proof of Theorem III.1

We consider the tangent map to the evaluation map $ev\colon P \times \mathcal{G} \to G$ at $(g, \chi) \in \mathcal{G} \ltimes \Omega^0(M, \operatorname{ad} P) \sim \mathrm{T}\mathcal{G}$.

It is a map

$$\mathrm{T}P \longrightarrow G \ltimes \mathfrak{g}$$

given by

$$\left(g(p), g^{-1}dg(p, X) + \chi(p)\right) \in G \ltimes \mathfrak{g},$$

where we have to remember that the logarithmic derivative $g^{-1}dg$ is a map defined on $\mathrm{T}P$ with values in $\mathfrak{g}$.

In order to show that it is a gauge transformation for $\mathrm{T}P$, it is enough to see that for any $(p, X) \in \mathrm{T}P$ and $(k, x) \in G \ltimes \mathfrak{g}$ we have the equation

$$g^{-1}dg\,\{(p, X)(k, x)\} = g^{-1}dg\,\{(pk, Xk + i(x)|_{pk})\} =$$

$$\operatorname{Ad}_{k^{-1}}[g^{-1}dg(p, X)] + x - \operatorname{Ad}_{k^{-1}}\operatorname{Ad}_{g^{-1}(p)}\operatorname{Ad}_k x = p_2\left(\operatorname{Ad}_{(k,x)^{-1}}\left\{g^{-1}dg(p, X)\right\}\right),$$

where $p_2 : \mathfrak{g} \oplus_s \mathfrak{g} \to \mathfrak{g}$ is the projection onto the second component. For any $\eta \in \Omega^1(M, \operatorname{ad} P)$ and for any $(g, \chi) \in \mathcal{G} \ltimes Lie(\mathcal{G})$ the map

$$(p, X) \rightsquigarrow \big(g(p), g^{-1}dg(p, X) + \chi(p) + \eta(p, X)\big)$$

is also a gauge transformation on $\mathrm{T}P$ and this shows that the inclusion $\mathrm{T}\mathcal{G} \hookrightarrow \mathcal{G}_{\mathrm{T}P}$ is proper. □

We have also shown that the group $\mathcal{G}_{\mathrm{T}P}$ includes the group $\mathcal{G}_{\mathrm{aff}}$ of *affine gauge transformations*, defined as the semidirect product $\mathrm{T}\mathcal{G} \ltimes \Omega^1(M, \operatorname{ad} P)$, whose elements are triples $(g, \chi, \eta) \in \mathcal{G} \ltimes \Omega^0(M, \operatorname{ad} P) \ltimes \Omega^1(M, \operatorname{ad} P)$. The product of two such triples is given by

$$(g_1, \chi_1, \tau_1)(g_2, \chi_2, \tau_2) = \Big(g_1 g_2, \mathrm{Ad}_{g_2^{-1}}\chi_1 + \chi_2, \mathrm{Ad}_{g_2^{-1}}\tau_1 + \tau_2,\Big)\,. \tag{III.11}$$

The Lie algebra of $\mathcal{G}_{\mathrm{aff}}$ is

$$\big(\Omega^0(M, \operatorname{ad} P) \oplus_s \Omega^0(M, \operatorname{ad} P)\big) \oplus_s \Omega^1(M, \operatorname{ad} P)$$

and the relevant commutator is given by

$$[(\zeta_1, \chi_1, \tau_1), (\zeta_2, \chi_2, \tau_2)] = ([\zeta_1, \zeta_2], [\zeta_1, \chi_2] + [\chi_1, \zeta_2], [\zeta_1, \tau_2] + [\tau_1, \zeta_2])\,.$$

IV. Tangent connections

Now we discuss the structure of the space of connections of the principal $\mathrm{T}G$-bundle $\mathrm{T}P$.

For any manifold M we can consider its double tangent bundle $\mathrm{TT}M$

$$\begin{array}{ccc} \mathrm{TT}M & \xrightarrow{\mathrm{T}\pi_M} & \mathrm{T}M \\ \big\downarrow{\scriptstyle \pi_{\mathrm{T}M}} & & \big\downarrow{\scriptstyle \pi_M} \\ \mathrm{T}M & \xrightarrow{\pi_M} & M. \end{array}$$

Let us consider a rectangle in M centered at x, i.e., a C^2-map $\underline{x}\colon [a, b] \times [a', b'] \mapsto M$, with $(0, 0) \in (a, b) \times (a', b')$, $\underline{x}(0, 0) = x$. We will denote by s and t the two variables and by a prime the derivative w.r.t. s, by a dot the derivative w.r.t. t. A tangent vector in $\mathrm{TT}M$ at $(\underline{x}(0, 0), \underline{x}'(0, 0))$ can be represented as $\left(\dot{\underline{x}}(0, 0), \dfrac{d\underline{x}'}{dt}(0, 0)\right)$. When it is not otherwise specified, the derivatives are understood to be computed at $(0, 0)$. Notice that two rectangles are meant to be equivalent if they have the same first order derivatives as well

as the mixed derivatives of order 2. Everything that follows is defined on the equivalence classes. Identifying double tangents with equivalence classes of rectangles provides us also with a (non-canonical) extension of the double tangent vector to the image of the rectangle.

In our notation we have, for a given $[\underline{x}] \in \mathrm{TT}M$, $\mathrm{T}\pi_M[\underline{x}] = \underline{\dot{x}}$ and $\pi_{\mathrm{T}M}[\underline{x}] = \underline{x}'$.

There is a canonical involution $\alpha_M \colon \mathrm{TT}M \mapsto \mathrm{TT}M$ given by

$$\alpha_M\big(x, \underline{x}', \underline{\dot{x}}, \frac{d\underline{x}'}{dt}\big) = \Big(x, \underline{\dot{x}}, \underline{x}', \frac{d\underline{\dot{x}}}{ds}\Big). \tag{IV.1}$$

It is well known [8] that for any connection A on P, one can induce a connection $\overset{\circ}{A}$ on $\mathrm{T}P$. If $\underline{p}$ is any rectangle in P centered at p, then $\overset{\circ}{A}$ is, by definition, the 1-form on $\mathrm{T}P$ with values in $\mathfrak{g} \oplus_s \mathfrak{g}$, given by

$$\overset{\circ}{A}_{p,\underline{p}'}\left(\underline{\dot{p}}, \frac{d\underline{p}'}{dt}\right) = \left(A_p\left(\underline{\dot{p}}\right), \left.\frac{d}{ds}\right|_{s=0}\left[A_{\underline{p}(s,0)}\left(\underline{\dot{p}}(s,0)\right)\right]\right). \tag{IV.2}$$

Notice that the canonical involution has been used in the definition of the above connection.

A more general connection on $\mathrm{T}P$ can be obtained by considering the evaluation map

$$ev \colon \mathcal{A} \times \mathrm{T}P \to \mathfrak{g}$$

$$(A; p, X) \rightsquigarrow A_p(X). \tag{IV.3}$$

The tangent map of (IV.3)

$$\mathrm{T}ev \colon \mathrm{T}\mathcal{A} \times \mathrm{TT}P \to \mathfrak{g} \oplus_s \mathfrak{g}, \tag{IV.4}$$

evaluated at $(A, \eta) \in \mathrm{T}\mathcal{A}$ and composed with the canonical involution α_P gives a connection on $\mathrm{T}P$. In fact we have the following

Theorem IV.1. *For any $(A, \eta) \in \mathrm{T}\mathcal{A}$, the 1-form $\omega(A, \eta)$ on $\mathrm{T}P$ given by*

$$\omega(A, \eta)([\underline{p}]) \equiv \mathrm{T}ev(A, \eta; \alpha_P[\underline{p}]) = \left(A_p\left(\underline{\dot{p}}\right), \quad \left.\frac{d}{ds}\right|_{s=0} A_{\underline{p}(s,0)}(\underline{\dot{p}}(s,0)) + \eta\left(\underline{\dot{p}}\right)\right) \tag{IV.5}$$

defines a connection on TP.

Proof of Theorem IV.1

The adjoint action of the group on its Lie algebra, extends, in an obvious way, to Lie-algebra valued forms. So we have a natural map $\mathrm{ad} \colon G \times \Omega^*(P, \mathfrak{g}) \mapsto \Omega^*(P, \mathfrak{g})$ and its derivative $\mathrm{T}\,\mathrm{ad}_G = \mathrm{ad}_{\mathrm{T}G}$ (see (III.4)). If we consider again the derivative in the first variable of the right multiplication on P, i.e., the map $\mathrm{T}_1 R : \mathrm{T}P \times G \to \mathrm{T}P$ and we denote by $\Delta_G : G \to G \times G$ the diagonal map in G and by $p_V : V \times U \to V$ the projection on V for any pair of spaces (V, U), then the ad-equivariance of the connections is given by the following equation

$$ev \circ p_{\mathcal{A} \times \mathrm{T}P} = ev \circ (\mathrm{ad} \times \mathrm{T}_1 R) \circ (\mathbb{I} \times \Delta_G) : \mathcal{A} \times \mathrm{T}P \times G \to \mathfrak{g}, \tag{IV.6}$$

where the evaluation map is given by (IV.3). The derivative of both sides of (IV.6) gives

$$\mathrm{T}ev \circ p_{\mathrm{T}\mathcal{A}\times\mathrm{TT}P} = \mathrm{T}ev \circ (\mathrm{ad}_{\mathrm{T}G} \times \mathrm{T}_1 R) \circ (\mathbb{I} \times \Delta_{\mathrm{T}G})\,. \tag{IV.7}$$

The equivariance of (IV.5) follows when we notice that

$$\mathrm{TT}_1 R = \alpha_P \circ \mathrm{T}_1 \mathrm{T}R \circ \alpha_P : \mathrm{TT}P \times \mathrm{T}G \to \mathrm{TT}P,$$

where α_P is the canonical involution. Finally we notice that the difference $\omega(A,\eta) - \overset{\circ}{A}$ evaluated at $[\underline{p}]$ gives $\eta(\underline{\dot{p}})$. But the projection $\mathrm{T}\pi_P$ applied to the fundamental vector field in $(p,X) \in \mathrm{T}P$ determined by $(a,b) \in \mathfrak{g} \oplus_s \mathfrak{g}$ yields $i(a)_p$, i.e., the fundamental vector field in P corresponding to a evaluated at p. The theorem follows from the fact that $\overset{\circ}{A}$ is a connection and η is a tensorial 1-form. □

When we consider the inclusion

$$\iota : P \hookrightarrow \mathrm{T}P, \quad \iota(p) = (p,0), \tag{IV.8}$$

then the pullback of the connection $\omega(A,\eta)$ via (IV.8) is simply given by

$$[\iota^*\omega(A,\eta)]_p(X) = (A(X)_p, \eta(X)_p) \in \mathfrak{g} \oplus_s \mathfrak{g}, \quad X \in \mathrm{T}_p P. \tag{IV.9}$$

Also we have the following

Theorem IV.2. *The inclusion of* $\mathrm{T}\mathcal{A}$ *into the space of connections of* $\mathrm{T}P$ *is proper.*

Proof of Theorem IV.2

The bundle $\mathrm{T}\mathcal{A}$ can be identified with an affine space modelled on

$$\Omega^1(M, \mathrm{ad}\,P) \oplus \Omega^1(M, \mathrm{ad}\,P) \sim \Omega^1(M, \mathrm{ad}\,P \oplus \mathrm{ad}\,P),$$

while the space of connections on $\mathrm{T}P$ can be identified with an affine space modelled on $\Omega^1(\mathrm{T}M, \mathrm{ad}\,\mathrm{T}P)$. The theorem is proved when we notice that the projection $M \to \mathrm{T}M$ determines a proper inclusion of $\Omega^*(M)$ into $\Omega^*(\mathrm{T}M)$ and that the pair of projections $\mathrm{T}P \to P$, $\mathrm{T}G \to G$ determines:

1. a homomorphism of the $\mathrm{T}G$-principal bundle $\mathrm{T}P \times \mathfrak{g} \times \mathfrak{g} \to \mathrm{ad}\,\mathrm{T}P$ (where the left $\mathrm{T}G$-adjoint action on $\mathfrak{g} \times \mathfrak{g}$ is considered) onto the G-principal bundle $P \times \mathfrak{g} \times \mathfrak{g} \to \mathrm{ad}\,P \oplus \mathrm{ad}\,P$ (where the left G-adjoint action on $\mathfrak{g} \times \mathfrak{g}$ is considered), and hence
2. a proper inclusion of the space of sections $\Gamma\,(\mathrm{ad}\,P \oplus \mathrm{ad}\,P)$ into the space of sections $\Gamma\,(\mathrm{ad}\,\mathrm{T}P)$. □

Two final remarks are in order.

A) By iterating the procedure, we obtain an inclusion

$$\mathrm{T}^n\mathcal{A}(P) \hookrightarrow \mathcal{A}(\mathrm{T}^n P)$$

where T^n denotes the n-iterated tangent bundle and $\mathcal{A}(\bullet)$ denotes the space of connections of a given principal bundle.

B) When we consider only irreducible connections, then $\mathcal{A} \mapsto \mathcal{A}/\mathcal{G}$ is a principal bundle and so is $\mathrm{T}^n\mathcal{A} \mapsto \mathrm{T}^n\mathcal{A}/\mathrm{T}^n\mathcal{G}$.

V. Topological field theories for tangent principal bundles

Here we want to extend the topological field theory discussed in section **II** to the case when the principal bundle is the *tangent bundle* $\mathrm{T}P$. The purpose of this section is to write the BRST equations for this case.

Here we will consider only those special connections for $\mathrm{T}P$ that are determined by elements of $\mathrm{T}\mathcal{A}$, i.e., by pairs (A,η) where A is a connection on P and η is any element of $\Omega^1(M, \mathrm{ad}\, P)$.

In analogy with the discussion of section **II** we will begin by including, among the admissible symmetries, besides the tangent gauge group $\mathrm{T}\mathcal{G}$, the full translation group on the affine space $\mathrm{T}\mathcal{A}$.

Namely, we want to consider the action of the group $\mathrm{T}\mathcal{G}_{\mathrm{T}}$ (tangent bundle of the group $\mathcal{G}_{\mathrm{T}}$ defined in (II.1)) on $\mathrm{T}\mathcal{A}$. This action is given by the derivative of (II.2)

$$\begin{gathered}\mathrm{T}\mathcal{A} \times \mathrm{T}\mathcal{G}_{\mathrm{T}} \to \mathrm{T}\mathcal{A},\\ (A,\eta)(g,\rho,\xi,\tau) \rightsquigarrow (A^g + \rho, \mathrm{Ad}_{g^{-1}}\,\eta + d_{A^g}\xi + \tau).\end{gathered} \tag{V.1}$$

Since (II.2) is not free, neither is (V.1).

We have an obvious infinite-dimensional analogue of Theorem IV.1:

Theorem V.1. *Any pair $(c,\hat{c})$ where c is a connection on (II.3) and $\hat{c}$ is a tangent vector to the space of connections on (II.3), defines a connection $\omega(c,\hat{c})$ on*

$$\mathrm{T}\mathcal{A} \mapsto \frac{\mathrm{T}\mathcal{A}}{\mathrm{T}\mathcal{G}}. \tag{V.2}$$

Explicitly $\hat{c}$ is an assignment to each connection $A \in \mathcal{A}$ of a map $\hat{c}_A \colon \Omega^1(M, \mathrm{ad}\, P) \mapsto \Omega^0(M, \mathrm{ad}\, P)$ with the property of $\mathcal{G}$-equivariance

$$\hat{c}_{A^g}\left(\mathrm{Ad}_{g^{-1}}\tau\right) = \mathrm{Ad}_{g^{-1}}\left(\hat{c}_A(\tau)\right),$$

and of tensoriality

$$\mathrm{Im}\left(d_A\big|_{\Omega^0(M,\mathrm{ad}\, P)}\right) \subset \ker(\hat{c}_A).$$

In physics $\hat{c}$ is an *infinitesimal variation of the gauge fixing.*

By using the same notation of the previous theorem, we have also the following

Theorem V.2. *The pair $(c,\hat{c})$ determines a connection on the* $\mathrm{T}G$ *principal bundle* $\mathrm{T}P \times \mathrm{T}\mathcal{A} \to \mathrm{T}M \times \mathrm{T}\mathcal{A}$ *that is* $\mathrm{T}\mathcal{G}$*-invariant, i.e., determines a connection on the principal* $\mathrm{T}G$*-bundle*

$$\frac{\mathrm{T}P \times \mathrm{T}\mathcal{A}}{\mathrm{T}\mathcal{G}} \mapsto \mathrm{T}M \times \frac{\mathrm{T}\mathcal{A}}{\mathrm{T}\mathcal{G}}. \tag{V.3}$$

If we consider a rectangle $\big([\underline{p}],[\underline{A}]\big)$ on $\mathrm{T}P\times\mathrm{T}\mathcal{A}$ centered at $(p,A)\in P\times\mathcal{A}$, with $X=\dot{\underline{p}}$, and $\eta=\dot{\underline{A}}$, then the explicit expression of such a connection at $(p,X,A,\eta)\in\mathrm{T}P\times\mathrm{T}\mathcal{A}$ is given by

$$\omega(A,\eta)([\underline{p}])+\omega(c,\hat{c})([\underline{A}]), \tag{V.4}$$

where the notation here is completely analogous to the one of Theorem IV.1.

A simple calculation shows

Theorem V.3. *When we identify the double tangent bundle* $\mathrm{TT}\mathcal{A}$ *with* $\mathcal{A}\times\Omega^1(M,\operatorname{ad}P)^{\times 3}$*, then the connection determined by* $(c,\hat{c})$ *is a map*

$$(A,\eta,\gamma,\sigma)\rightsquigarrow\left(c_A(\gamma),\,\frac{d}{ds}\bigg|_{s=0}c_{A+s\eta}(\gamma)+\hat{c}_A(\gamma)+c_A(\sigma)\right).$$

From now on we set

$$\xi_{A,\eta}(\gamma)\equiv\frac{d}{ds}\bigg|_{s=0}c_{A+s\eta}(\gamma)+\hat{c}_A(\gamma). \tag{V.5}$$

Since space–time for us is M and not $\mathrm{T}M$, we want to avoid dealing with forms on $\mathrm{T}P$, as opposed to forms on P.

We consider then again the inclusion map (IV.8) $P\to\mathrm{T}P$ and establish the following

Convention i. *All the forms on* $\mathrm{T}P\times\mathrm{T}\mathcal{A}$ *are pulled back to forms on* $P\times\mathrm{T}\mathcal{A}$*.*

In the use of Convention **i** we should be aware that, even though (IV.8) is a morphism of principal bundles, the pullback of a connection on $\mathrm{T}P\times\mathrm{T}\mathcal{A}$ is *not* (strictly speaking) a connection on $P\times\mathrm{T}\mathcal{A}$. It is a $(\mathfrak{g}\oplus_s\mathfrak{g})$-valued 1-form that would become a connection only if we decided to disregard the second copy of $\mathfrak{g}$.

Moreover, when considering the decomposition $\mathrm{T}\mathcal{A}\sim\mathcal{A}\times\Omega^1(M,\operatorname{ad}P)$ we also assume the following

Convention ii. *We will omit the components of the forms on* $P\times\mathrm{T}\mathcal{A}$ *which have degree higher than 0 as forms on* $\Omega^1(M,\operatorname{ad}P)$*.*

When we assume both the previous conventions, then the connection (V.4) becomes the following $(\mathfrak{g}\oplus_s\mathfrak{g})$-valued 1-form on $P\times\mathcal{A}$, depending on $\eta\in\Omega^1(M,\operatorname{ad}P)$

$$(A+c;\ \eta+\xi)_{p,A,\eta}(X,\gamma)\equiv\left(A_p(X)+c_A(\gamma);\ \eta_p(X)+\xi_{A,\eta}(\gamma)\right),\quad X\in\mathrm{T}_pP,\gamma\in\mathrm{T}_A\mathcal{A}, \tag{V.6}$$

where (V.5) has been used.

We will, in the future, refer to (V.6) as a "connection" with an associate "covariant exterior derivative," "curvature" and "Bianchi identities;" but, as has been explained above, this will clearly be an abuse of language.

The covariant exterior derivative for $(\mathfrak{g} \oplus_s \mathfrak{g})$-valued forms on $P \times \mathcal{A}$ is given by

$$d_{(A,\eta)} + \pm\delta_{(c,\xi)}. \tag{V.7}$$

More precisely the covariant exterior derivative applied to the pair of $\mathfrak{g}$-valued forms (ω, ω'), with degree (k, k') as forms on P, is given by

$$\Big(d_A\omega + (-1)^k\delta\omega + (-1)^k[c,\omega], d_A\omega' + (-1)^{k'}\delta\omega' + [\eta + (-1)^k\xi, \omega] + [(-1)^{k'}c, \omega']\Big) \tag{V.8}$$

where we used (III.5). The curvature of (V.6) is a $(\mathfrak{g} \oplus_s \mathfrak{g})$-valued 2-form on $P \times \mathcal{A}$ which we can compute in an similar way as

$$(F_A + \psi + \phi; \quad d_A\eta + \tilde{\psi} + \tilde{\phi}), \tag{V.9}$$

where $(F_A, d_A\eta)$, $(\psi, \tilde{\psi})$ and $(\phi, \tilde{\phi})$ are respectively the $(2,0)$, the $(1,1)$ and the $(0,2)$ components.

We now compute explicitly the structure equation and the Bianchi identities.

First we obtain once again the equations (II.8), (II.9) and (II.10).

The structure equations give moreover

$$\tilde{\psi} = -\delta\eta + d_A\xi + [\eta, c], \tag{V.10}$$

$$\tilde{\phi} = \delta\xi + [c, \xi]. \tag{V.11}$$

Finally the Bianchi identities for the *second* component of the curvature give

$$\delta\tilde{\psi} = -[\tilde{\psi}, c] + d_A\tilde{\phi} + [\eta, \phi] - [\xi, \psi], \quad \delta\tilde{\phi} = [\tilde{\phi}, c] + [\phi, \xi]. \tag{V.12}$$

So we have:

Theorem V.4. *The transformations for the set of fields $A, \eta, c, \xi, \psi, \phi, \tilde{\psi}, \tilde{\phi}$ given by the components of the connection (V.6) and the curvature (V.9) are as follows*

$$\delta A = d_A c - \psi \tag{V.13}$$

$$\delta\eta = -\tilde{\psi} + d_A\xi + [\eta, c] \tag{V.14}$$

$$\delta F_A = -d_A\psi + [F_A, c] \tag{V.15}$$

$$\delta(d_A\eta) = -d_A\tilde{\psi} + [F_A, \xi] + [d_A\eta, c] - [\psi, \eta] \tag{V.16}$$

$$\delta c = -\frac{1}{2}[c, c] + \phi \tag{V.17}$$

$$\delta\xi = \tilde{\phi} - [c, \xi] \tag{V.18}$$

$$\delta\psi = d_A\phi - [\psi, c] \tag{V.19}$$

$$\delta\tilde{\psi} = -[\tilde{\psi}, c] + d_A\tilde{\phi} + [\eta, \phi] - [\xi, \psi] \tag{V.20}$$

$$\delta\phi = [\phi, c] \tag{V.21}$$

$$\delta\tilde{\phi} = [\tilde{\phi}, c] + [\phi, \xi]. \tag{V.22}$$

The transformation laws in Theorem V.4 show how to overcome the problem of the lack of freedom of (V.1). As before the key element has been the introduction of a (equivariant) coboundary operator relevant to a principal bundle (in this case (V.3)).

VI. Restrictions to the orbits

Two possible fiber imbeddings can be considered:

$$j_A : \mathcal{G} \hookrightarrow \mathcal{A}, \quad j_A(g) = A^g, \tag{VI.1}$$

$$j_{(A,\eta)} : \mathrm{T}\mathcal{G} \hookrightarrow \mathrm{T}\mathcal{A}, \quad j_{(A,\eta)}(g,\chi) = \left(A^g, \mathrm{Ad}_{g^{-1}}(\eta) + d_{A^g}\chi\right). \tag{VI.2}$$

First we can pull back the bundle

$$P \times \mathrm{T}\mathcal{A} \mapsto M \times \mathrm{T}\mathcal{A} \tag{VI.3}$$

to

$$P \times \mathrm{T}\mathcal{G} \mapsto M \times \mathrm{T}\mathcal{G}$$

via (VI.2).

Then we pull back the 1-form (c,ξ) (without changing notation). This form then becomes the Maurer–Cartan form on $\mathrm{T}\mathcal{G}$, and the structure equations and Bianchi identities become simply

$$\delta A = d_A c, \quad \delta c = -\frac{1}{2}[c,c], \quad \delta\xi = -[c,\xi], \quad \delta\eta = d_A\xi + [\eta,c]. \tag{VI.4}$$

This is the generalization to the tangent bundle of the ordinary BRST equations [7].

Starting from (VI.1) we obtain another restriction that will allow us to have a BRST symmetry that includes the 2-form field B considered in 4-dimensional BF theories.

We begin by noticing that the group $\mathcal{G}_{\mathrm{aff}}$ considered in section **III** is the right symmetry group for the action functional (I.4). Here the action of $\mathcal{G}_{\mathrm{aff}}$ on triples (A,η,B) with $(A,\eta) \in \mathrm{T}\mathcal{A}$ and $B \in \Omega^2(M, \mathrm{ad}\,P)$ is defined as follows[1]

$$(A,\eta)\cdot(g,\zeta,\tau) = \left(A^g, \tau + \mathrm{Ad}_{g^{-1}}\eta + d_{A^g}\zeta\right), \tag{VI.5}$$

$$B\cdot(g,\zeta,\tau) = \mathrm{Ad}_{g^{-1}} B + d_{A^g}\tau + d^2_{A^g}\zeta. \tag{VI.6}$$

The group $\mathcal{G}_{\mathrm{aff}}$ is just a subgroup of $\mathrm{T}\mathcal{G}_{\mathrm{T}}$ since we have

$$\mathcal{G}_{\mathrm{aff}} \equiv j^*(\mathrm{T}\mathcal{G}_{\mathrm{T}}),$$

where j is the inclusion $j : \mathcal{G} \hookrightarrow \mathcal{G}_{\mathrm{T}}$. The action (VI.5) is the corresponding restriction of (V.1).

Requiring the invariance under the group of affine gauge transformations is tantamount to considering a *"semi-topological" field theory where both the invariance under* $\mathrm{T}\mathcal{G}$ *and the independency of the choice of the tangent vector* $\eta \in \mathrm{T}_A\mathcal{A}$ *is considered.*

[1] Notice that this action of $\mathcal{G}_{\mathrm{aff}}$ on $\mathrm{T}\mathcal{A}$ does *not* coincide with the restriction of the action of $\mathcal{G}_{\mathrm{T}P}$ on $\mathcal{A}(\mathrm{T}P)$.

To turn around the problem of the non freedom of $(VI.5)$ we proceed along the same lines of the previous section.

First we consider the pulled-back bundle $j_A^*\mathrm{T}\mathcal{A}$ and pull back to it the 1-form (c, ξ) (again without changing notation).

Now, as in (VI.4), c becomes the Maurer–Cartan form on $\mathcal{G}$, but ξ, differently from (VI.4), does not become any more the second component of the Maurer–Cartan form on $\mathrm{T}\mathcal{G}$. The form ξ is defined as in (V.5), provided that the connection on P belongs to the $\mathcal{G}$-orbit passing through A.

The corresponding pulled-back "connection" on the bundle

$$P \times j_A^*\mathrm{T}\mathcal{A} \mapsto M \times j_A^*\mathrm{T}\mathcal{A}, \tag{VI.7}$$

is written again as

$$(A + c, \eta + \xi) \tag{VI.8}$$

and the corresponding "curvature" is

$$(F_A, d_A\eta + \tilde{\psi} + \tilde{\phi}). \tag{VI.9}$$

The Bianchi identities become

$$[d_{(A,\eta)} \pm \delta_{(c,\xi)}](F_A, d_A\eta + \tilde{\psi} + \tilde{\phi}) = 0$$

and altogether we obtain (see [1]):

Theorem VI.1. *The transformation laws for the set of fields $A, \eta, c, \xi, \tilde{\psi}, \tilde{\phi}$ given by the components of the connection (VI.8) and of the curvature (VI.9) are as follows:*

$$\delta A = d_A c \tag{VI.10}$$

$$\delta\eta = -\tilde{\psi} + d_A\xi + [\eta, c] \tag{VI.11}$$

$$\delta F_A = [F_A, c] \tag{VI.12}$$

$$\delta(d_A\eta) = -d_A\tilde{\psi} + [F_A, \xi] + [d_A\eta, c] \tag{VI.13}$$

$$\delta c = -\frac{1}{2}[c, c] \tag{VI.14}$$

$$\delta\xi = \tilde{\phi} - [c, \xi] \tag{VI.15}$$

$$\delta\tilde{\psi} = -[\tilde{\psi}, c] + d_A\tilde{\phi} \tag{VI.16}$$

$$\delta\tilde{\phi} = [\tilde{\phi}, c]. \tag{VI.17}$$

By comparing (VI.10), (VI.11) with (VI.5) we see that also this last BRST operator δ allowed us to overcome the problem of the lack of freedom of (VI.5). We have still to find how to read in our BRST complex the transformations of the field B corresponding to the action (VI.6). This will accomplished in the next section.

VII. A BRST complex that includes 2-forms

Now we will show that the BRST operator δ can consistently be extended to an operator s satisfying $s^2 = 0$ and that a double complex with operators (d, s) can be constructed with the following properties:
1. s acts on the space $\Omega^2(M, \mathrm{ad}\, P)$;
2. the gauge-equivariance is preserved, and
3. the equations considered in Theorem VI.1 are preserved.

We use the synthetic notation $\mathcal{B} \triangleq \Omega^2(M, \mathrm{ad}\, P)$ and consider the tangent bundle $\mathrm{T}\mathcal{B} \sim \Omega^2(M, \mathrm{ad}\, P) \times \Omega^2(M, \mathrm{ad}\, P)$.

The group $\mathcal{G}$ acts on $\mathcal{A} \times \mathcal{B}$ yielding a $\mathcal{G}$-principal bundle. Moreover, the tangent map to the above action gives $\mathrm{T}\mathcal{A} \times \mathrm{T}\mathcal{B}$ the structure of a $\mathrm{T}\mathcal{G}$-principal bundle.

Explicitly this last action is given by

$$(A, \eta; C, E) \cdot (g, \zeta) =$$

$$\left(A^g, \mathrm{Ad}_{g^{-1}}\eta + d_{A^g}\zeta; \mathrm{Ad}_{g^{-1}}C, \mathrm{Ad}_{g^{-1}}E + [\mathrm{Ad}_{g^{-1}}C, \zeta]\right), \qquad \text{(VII.1).}$$

It is evident that the projection $\mathcal{A} \times \mathcal{B} \mapsto \mathcal{A}$ is a morphism of $\mathcal{G}$-bundles. Therefore, the connection c on (II.3) can also be considered as a connection on $\mathcal{A} \times \mathcal{B}$, and the connection (c, ξ) on (V.2) is also a connection on $\mathrm{T}\mathcal{A} \times \mathrm{T}\mathcal{B}$.

Moreover, $(A + c, \eta + \xi)$ is a (pulled-back) $(\mathfrak{g} \oplus_s \mathfrak{g})$-valued connection on the bundle

$$P \times \mathrm{T}\mathcal{A} \times \mathrm{T}\mathcal{B} \mapsto M \times \mathrm{T}\mathcal{A} \times \mathrm{T}\mathcal{B}. \qquad \text{(VII.2)}$$

The corresponding covariant exterior derivative is given by (V.7). Forms on $P \times \mathrm{T}\mathcal{A} \times \mathrm{T}\mathcal{B}$ will be characterized by three indices (m, s, p) which represent the degree with respect to the three spaces P, $\mathrm{T}\mathcal{A}$, $\mathrm{T}\mathcal{B}$. The integer s is, as before, called the *ghost number*.

The pair $(C, E) \in \mathrm{T}\mathcal{B}$ is a $(\mathfrak{g} \oplus_s \mathfrak{g})$-valued $(2, 0, 0)$-form that is independent of $\mathrm{T}\mathcal{A}$.

We assume Convention **i** and Convention **ii**, and also the following

Convention iii. *We will omit the components of the forms on $P \times \mathrm{T}\mathcal{A} \times \mathrm{T}\mathcal{B}$ which have degree higher than 0 as forms on $\mathrm{T}\mathcal{B}$.*

Under the action of the covariant exterior derivative (V.7), the pair (C, E) is transformed into

$$(d_A C + [c, C], d_A E + [\eta, C] + [c, E] + [\xi, C]), \qquad \text{(VII.3)}$$

owing to the fact that (C, E) does not depend on the space $\mathcal{A}$.

Definition VII.1. *We define s to be equal to δ when computed on forms over $P \times \mathrm{T}\mathcal{A}$; moreover, we set for $(C, E) \in \mathrm{T}\mathcal{B}$.*

$$s(C, E) \triangleq ([C, c], [C, \xi] + [E, c]). \qquad \text{(VII.4)}$$

We have hence set $s(C,E)$ to be equal, up to a sign, to $\delta_{(c,\xi)}(C,E)$. This takes care of requirement 2 (preservation of gauge-equivariance).

As a consequence we have

$$\delta^2_{(c,\xi)}(C,E) = [(\phi,\tilde{\phi}),(C,E)] = ([\phi,C],[\tilde{\phi},C]+[\phi,E]),$$

where $(\phi,\tilde{\phi})$ is the curvature of the connection (c,ξ) on the $\mathrm{T}\mathcal{G}$ bundle $\mathrm{T}\mathcal{A}$. Here Convention **i** and Convention **ii** are always assumed.

In conclusion we have:

Theorem VII.2. *Let us assume Definition VII.1. The operator s is nilpotent in the following cases:*

1. *always in the case of equations (VI.4);*
2. *never in the case of the equations of Theorem V.4;*
3. *only when pairs $(0,E)\in \mathrm{T}_0\mathcal{B}$ are considered in the case of the equations of Theorem VI.1.*

Remark VII.3. *The reason why, when we consider the bundle (VII.2), we have non-trivial solutions for the compatibility problem of Theorem VII.2, is the fact that $\{1\}\times\mathfrak{g}\triangleleft G\ltimes\mathfrak{g}$ in an abelian normal subgroup.*

We now translate the field E and define

$$B \equiv E + d_A\eta. \tag{VII.5}$$

By combining (VI.13) with (VII.4) we find the following transformation property of B (see [1] and [9])

$$sB = -d_A\tilde{\psi} + [F_A,\xi] + [B,c]. \tag{VII.6}$$

Equation (VII.6) may be added to equations (VI.10) $\longrightarrow$ (VI.17), namely,

$$sA = d_Ac, \quad s\eta = -\tilde{\psi} + d_A\xi + [\eta,c],$$

$$sc = -\frac{1}{2}[c,c], \quad s\xi = \tilde{\phi} - [c,\xi],$$

$$s\tilde{\psi} = -[\tilde{\psi},c] + d_A\tilde{\phi}, \quad s\tilde{\phi} = [\tilde{\phi},c].$$

While E is an element of $\mathrm{T}_0\mathcal{B}$, B is naturally an element of $\mathrm{T}_{F_A}\mathcal{B}$. In particular, triples (A,η,B) are elements of the pulled-back bundle

$$K^*(\mathrm{T}\mathcal{A}\times\mathrm{T}\mathcal{B}),$$

where K is the curvature map

$$K:\mathcal{A}\mapsto\mathcal{A}\times\mathcal{B}, \quad K(A)\triangleq(A,F_A). \tag{VII.7}$$

As a final remark, notice that once again we have overcome the problem of the lack of freedom of (VI.6).

VIII. Infinite BRST symmetries

The procedure of the previous two sections can be iterated, so as to incorporate an arbitrary number of fields. More precisely, given any integer n we can construct field equations, similar to those considered in the previous section, depending on a connection A, on $2^n - 1$ fields $\gamma_i \in \Omega^1(M, \operatorname{ad} P)$, on 2^n fields $\eta_i \in \Omega^1(M, \operatorname{ad} P)$, on 2^n fields $B_i \in \Omega^2(M, \operatorname{ad} P)$, and on the relevant ghost-fields.

A binary notation turns out to be helpful to describe iterated tangent spaces. In fact given a manifold M we will describe an element $\mathbf{X} \in \mathrm{T}^n M$ by a 2^n-tuple as follows

$$\mathbf{X} = \left(X_{\underbrace{0\dots00}_{n-\text{times}}}, X_{\underbrace{0\dots01}_{n-\text{times}}}, X_{\underbrace{0\dots10}_{n-\text{times}}}, \dots, X_{\underbrace{1\dots11}_{n-\text{times}}} \right).$$

For instance in the case $n = 3$, X_{000} denotes a point of M, (X_{000}, X_{001}) denotes a point in $\mathrm{T}M$, $(X_{000}, X_{001}, X_{010}, X_{011})$ denotes a point in $\mathrm{TT}M$ whose tangent vector is represented by $(X_{100}, X_{101}, X_{110}, X_{111})$.

Elements in $Lie(\mathrm{T}^n G)$ can analogously be represented as 2^n-tuples $\boldsymbol{\xi} = (\xi_{i_1,\dots,i_n})$ where each index i_p can be either 0 or 1, $\forall p \in \{1, \dots, n\}$.

We now want to describe the bracket in $Lie(\mathrm{T}^n G)$ induced by the bracket in $\mathfrak{g}$. We have

Theorem VIII.1. *The Lie bracket of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ in $Lie(\mathrm{T}^n G)$ is given by:*

$$[\boldsymbol{\xi}, \boldsymbol{\eta}]_{i_1 \cdots i_n} = \sum_{j_i + k_i = i_i} [\xi_{j_1 \cdots j_n}, \eta_{k_1 \cdots k_n}] \tag{VIII.1}$$

where the previous notation is understood.

Proof of Theorem VIII.1

We will prove (VIII.1) by induction on n. Formula (VIII.1) holds for $n = 0$, i.e., for $\mathfrak{g}$. We assume now that it is true for n. We can represent any element $\boldsymbol{\xi}$ of $Lie(\mathrm{T}^{n+1} G)$ as $(\xi_{0 j_1 \cdots j_n}, \xi_{1 j_1 \cdots j_n})$ where the first component represents the base point in $Lie(\mathrm{T}^n G)$, while the second component represents the relevant tangent vector. We can therefore use the expression of the bracket for $Lie(\mathrm{T}G) \sim \mathfrak{g} \oplus_s \mathfrak{g}$ thus getting

$$[(\xi_{0 j_1 \dots j_n}, \xi_{1 j_1 \dots j_n}), (\eta_{0 k_1 \cdots k_n}, \eta_{1 k_1 \dots k_n})] = \Big([\xi_{0 j_1 \dots j_n}, \eta_{0 k_1 \dots k_n}],$$

$$[\xi_{0 j_1 \dots j_n}, \eta_{1 k_1 \dots k_n})] + [\xi_{1 j_1 \dots j_n}, \eta_{0 k_1 \dots k_n})] \Big)$$

Therefore,

$$[\boldsymbol{\xi}, \boldsymbol{\eta}]_{i_0 i_1 \ldots i_n} = \sum_{j_i + k_i = i_i} [\xi_{j_0 j_1 \ldots j_n}, \eta_{k_0 k_1 \ldots k_n}]$$

□

We now take into account the iterated gauge group $\mathrm{T}^n\mathcal{G}$.

We can consider the Maurer–Cartan form for $\mathrm{T}^n\mathcal{G}$. This is a form with values in $Lie(\mathrm{T}^n\mathcal{G})$, so it can be represented by a vector $\mathbf{c}$ with 2^n components each having ghost number one.

We set $c_0 \equiv c_{0\ldots 0}$, and $c_I \equiv c_{i_1 \ldots i_n}$ for $I = \{i_1, \ldots, i_n\} \neq \{0 \ldots 0\}$, and define $I + J \triangleq \{i_1 + j_1, \ldots, i_n + j_n\}$ (*not* modulo 2).

The *Maurer–Cartan* equations are as follows:

$$\delta c_0 = -\frac{1}{2}[c_0, c_0], \tag{VIII.2}$$

$$\delta c_I = -\frac{1}{2} \sum_{J+K=I} [c_J, c_K]. \tag{VIII.3}$$

The group $\mathrm{T}^n\mathcal{G}$ acts (freely) on the iterated tangent space $\mathrm{T}^n\mathcal{A}$. We denote the corresponding fiber imbedding by

$$j^{(n)}_{A,\boldsymbol{\gamma}} \colon \mathrm{T}^n\mathcal{G} \to \mathrm{T}^n\mathcal{A},$$

where $(A, \boldsymbol{\gamma})$ represents an element of $\mathrm{T}^n\mathcal{A}$ with $\boldsymbol{\gamma} \in \Omega^1(M, \mathrm{ad}\, P)^{\times 2^n - 1}$.

In this section we consistently apply Convention **i**, Convention **ii** and Convention **iii**. The equations describing the infinitesimal action of $\mathrm{T}^n\mathcal{G}$ and corresponding to (VI.4) are as follows:

$$\delta A = d_A c_0 \tag{VIII.4}$$

$$\delta \gamma_I = d_A c_I + \sum_{J+K=I} [\gamma_J, c_K]. \tag{VIII.5}$$

In analogy with (VI.7) we apply the tangent functor once more and consider the bundle

$$P \times \left(j^{(n)}_{A,\boldsymbol{\gamma}}\right)^* \mathrm{T}(\mathrm{T}^n\mathcal{A}) \to M \times \left(j^{(n)}_{A,\boldsymbol{\gamma}}\right)^* \mathrm{T}(\mathrm{T}^n\mathcal{A}). \tag{VIII.6}$$

We have a vector $\boldsymbol{\eta}$ with 2^n components in $\Omega^1(M, \mathrm{ad}\, P)$ with ghost number zero and a vector $\boldsymbol{\xi}$ with 2^n components in $\Omega^0(M, \mathrm{ad}\, P)$ with ghost number one. In analogy with (VI.8) on (VIII.6) we have a (pulled-back) connection[2]

$$((A, \boldsymbol{\gamma}) + \mathbf{c}, \boldsymbol{\eta} + \boldsymbol{\xi}) \tag{VIII.7}$$

[2] We have to keep in mind that each of the four vectors $(A, \boldsymbol{\gamma}); \boldsymbol{\eta}; \mathbf{c}; \boldsymbol{\xi}$ is a form on $P \times \mathrm{T}^n\mathcal{A}$ with values in $Lie(\mathrm{T}^nG)$.

with values in $Lie(\mathrm{T}^{n+1}G)$.

The curvature of (VIII.7) is given by

$$\left(F_A + d_A\boldsymbol{\gamma} + \frac{1}{2}[\boldsymbol{\gamma},\boldsymbol{\gamma}], d_A\boldsymbol{\eta} + [\boldsymbol{\gamma},\boldsymbol{\eta}] + \tilde{\boldsymbol{\psi}} + \tilde{\boldsymbol{\phi}}\right), \tag{VIII.8}$$

where the I-th component of the vector $[\boldsymbol{\gamma},\boldsymbol{\eta}]$ is defined to be $\sum_{J+K=I}[\gamma_J,\eta_K]$, the vector $\tilde{\boldsymbol{\psi}}$ has 2^n components in $\Omega^1(M,\mathrm{ad}\,P)$ and ghost number one, while the vector $\tilde{\boldsymbol{\phi}}$ has 2^n components respectively in $\Omega^0(M,\mathrm{ad}\,P)$ and ghost number two.

The results of the previous section, suggest us to introduce a vector $\mathbf{B}$ with 2^n components in $\Omega^2(M,\mathrm{ad}\,P)$ and ghost number zero.

By performing calculations similar to those carried on in the proof of Theorem VI.1 and (VII.6), we obtain the following

Theorem VIII.2. *The transformations for the set of fields $A,\boldsymbol{\gamma},\boldsymbol{\eta},\mathbf{c},\boldsymbol{\xi},\tilde{\boldsymbol{\psi}},\tilde{\boldsymbol{\phi}},\mathbf{B}$ are given by (VIII.2), (VIII.3), (VIII.4), (VIII.5) and*

$$\delta\eta_I = -\tilde{\psi}_I + d_A\xi_I + \sum_{J+K=I}[\gamma_J,\xi_K] + \sum_{J+K=I}[\eta_J,c_K] \tag{VIII.9}$$

$$\delta B_I = -d_A\tilde{\psi}_I - \sum_{J+K=I}[\gamma_J,\tilde{\psi}_K] + [F_A,\xi_I] + \sum_{J+K=I}[d_A\gamma_J,\xi_K] + \sum_{J+K=I}[B_J,c_K] \tag{VIII.10}$$

$$\delta\xi_I = \tilde{\phi}_I - \sum_{J+K=I}[c_J,\xi_K] \tag{VIII.11}$$

$$\delta\tilde{\psi}_I = -\sum_{J+K=I}[\tilde{\psi}_J,c_K] + d_A\tilde{\phi}_I + \sum_{J+K=I}[\gamma_J,\tilde{\phi}_K] \tag{VIII.12}$$

$$\delta\tilde{\phi}_I = \sum_{J+K=I}[\tilde{\phi}_J,c_K]. \tag{VIII.13}$$

IX. The Batalin Vilkovisky alternative: topological BF theories

In sections **VI** and **VII** we have considered the BRST structure of a field theory with fields $(A,\eta)\in \mathrm{T}\mathcal{A}$ and $B\in\mathcal{B}$ that is gauge invariant and can be considered *topological* only as far as the translations of the field η are concerned. Equivalently such a theory admits, as its basic symmetry group, the group $\mathcal{G}_{\mathrm{aff}}$.

For a full topological BF field theory, where translations in the space $\mathcal{A}$ have to be included as symmetries, one should consider instead the action of the group $\mathcal{G}_{\mathrm{T}}$ on $\mathcal{A}$ given by (II.2) and extended to $\mathcal{A} \times \mathcal{B}$ as follows:

$$(A, B) \cdot (g, \tau) = \left(A^g + \tau, \mathrm{Ad}_{g^{-1}} B - d_{A^g} \tau - \frac{1}{2}[\tau, \tau] \right). \tag{IX.1}$$

Notice that (IX.1) implies that the action of $\mathcal{G}_{\mathrm{T}}$ on $B + F_A$ is simply the adjoint action of $\mathcal{G}$.

An example of such a theory is given by the action functional (I.2) when $t = 1$, i.e., by

$$S_{BF,1} = \int_M \mathrm{Tr} \left(F_A \wedge B + \frac{1}{2} B \wedge B \right). \tag{IX.2}$$

Again the action (IX.1) is not free, but now this problem cannot be solved by considering the coboundary (BRST) operator of a principal bundle.

In fact one would have to extend the coboundary operator δ defined on (II.4) to a nilpotent operator s acting on B, while preserving the gauge equivariance.

This would mean to consider a covariant derivative acting on $\mathcal{B}$ corresponding to a connection c on $\mathcal{A}$. The requirement of the nilpotency of such an operator would then be equivalent to the requirement that, for any $B \in \mathcal{B}$, the following condition should hold:

$$[\phi, B] = 0, \tag{IX.3}$$

where ϕ is the curvature of c.

But we know that condition (IX.3) is not satisfied.

The so called Batalin–Vilkovisky mechanism [10] now comes into the picture by providing us with a nilpotent coboundary operator that has the same properties of a coboundary operator for a principal bundle.

The price we have to pay is that we have to introduce fields with negative ghost-number or, equivalently, that we have to consider fields that depend explicitly on the (functional integral with respect to the) action that we are considering.

The functional integral *formally* provides us with a volume form on the space of fields

$$\omega \equiv \exp(-S) * 1 \equiv \exp(-S) \prod_x \mathcal{D}A(x)\, \mathcal{D}B(x), \tag{IX.4}$$

where S is the action functional (IX.2). The understanding that such a volume form exists is at the heart of the Batalin–Vilkovisky mechanism, which is the same to say that the Batalin Vilkovisky mechanism is a device that allows *integration by parts in field theories.*

Let us denote by $\mathfrak{F}$ the "manifold" given by $\prod_j \prod_{x \in M} \mathcal{F}_j(x)$ where $\{\mathcal{F}_j\}$ is the set of all fields of the theory (e.g., $A^b, B^c, c^a, \psi^d, \phi^e$, with c, ψ, ϕ being defined as in section **II**) and the indices a, b, c, d, e labelling a basis for the Lie algebra $\mathfrak{g}$. We work here in local coordinates as far as the bundle $P(M, G)$ is concerned.

We denote by the symbol B^* the ($\mathfrak{g}$-valued) contraction with respect to the vector field corresponding to the coordinate $B^a(x)$, $x \in M$ of the "manifold" $\mathfrak{F}$. Namely, we set

$$(B^*)^a \triangleq \frac{\partial}{\partial B^a(x)}. \tag{IX.5}$$

Being the ghost number defined as the degree of a form on $\mathfrak{F}$ (and particularly on the space of connections $\mathcal{A}$), a vector field on $\mathfrak{F}$ will have ghost number -1. More generally, given a field σ with form degree k (on M) and ghost number s, its antifield σ^* will have form degree $4-k$ and ghost number $-s-1$. The duality between the σ and σ^* will be given by the following condition:

$$\{(\sigma^*)^a(x), \sigma^b(y)\} = \delta^a_b\, \delta(x-y),$$

where the Dirac volume form δ on M has been used, and $\{\bullet, \bullet\}$ denotes the Poisson bracket for the cotangent bundle $\mathrm{T}^*\mathfrak{F}$, which is mapped onto the tangent bundle via the metric (which gives the $*$-operator).

We now denote by δ the exterior derivative on $\mathfrak{F}$. By definition of B^* we have

$$\delta\left(B^*(*1)\right) = 0$$

and so

$$\delta(B^*\omega) = (-B(x) - F_A(x))\omega;$$

in other words we have the formal equation

$$\delta(B^*) = -B - F_A.$$

If we require equivarance, i.e., if we want to take into account the action of the group of gauge transformations $\mathcal{G}$, then we have to replace the exterior derivative with the covariant exterior derivative which we denote again by the symbol s. So we have

$$s(B^*) = -B - F_A - [B^*, c],$$

and the ensuing transformation

$$sB = [B, c] + d_A\psi - [B^*, \phi]. \tag{IX.6}$$

It is interesting to compare (IX.6) with (VII.6). The main difference between the two equations is that, in the second case, the operator s applied to B produces *only fields*, while it produces *both fields and antifields* in the first case. In the BV mechanism, antifields are added so to render the operator s nilpotent, i.e., so as to formally reconstruct a BRST operator. This is the case of the topological BF theory. On the contrary, in the BFYM theory, antifields are unnecessary since the BRST operator is nilpotent *per se.*

From the previous equations we have

$$\int [(B + F_A)\,\omega]\,\mathcal{D}A\,\mathcal{D}B\; = 0,$$

and in particular,

$$\frac{\partial \exp S}{\partial B} \approx 0,$$

$$F_A \exp(-S) = \exp\left(-\frac{1}{2}\int_M \mathrm{Tr}\,(B\wedge B)\right) B^* \left[\exp\left(-\int_M \mathrm{Tr}\,(B\wedge F_A)\right)\right] \approx$$
$$-\exp\left(-\int_M \mathrm{Tr}\,(B\wedge F_A)\right) B^* \left[\exp\left(-\frac{1}{2}\int_M \mathrm{Tr}\,(B\wedge B)\right)\right] = -B\exp(-S),$$

where $\approx$ means up to a total derivative. When the action functional is the topological BF action functional (IX.2), then $B^*(x)$ is equivalent to the transformation (concentrated at x)

$$F_A(x) \rightsquigarrow -B(x). \tag{IX.7}$$

We now include the fields (with negative ghost number) A^*, c^*, ψ^*, ϕ^* conjugate respectively to A, c, ψ, ϕ. The BV equations are written in the following double-complex

4				$sc^*=[c^*,c]-d_A A^*+$ $[B^*,B]-[\psi^*,\psi]-[\phi^*,\phi]$	$c^*=s\phi^*+[\phi^*,c]$ $+d_A\psi^*+\frac{1}{2}[B^*,B^*]$	ϕ^*
3			$sA^*+[A^*,c]=-d_A B$ $-[B^*,\psi]+[\psi^*,\phi]$	$A^*=-s\psi^*+$ $+[\psi^*,c]+d_A B^*$	ψ^*	
2		$sB+[c,B]=$ $=d_A\psi+[B^*,\phi]$	$-B=sB^*$ $+[B^*,c]+F_A$	B^*		
1	$-s\psi-[c,\psi]$ $=-d_A\phi$	$\psi=d_A c-sA$	A			
0	$\phi=sc$ $+\frac{1}{2}[c,c]$	c				
	2	1	0	-1	-2	-3
	$\xleftarrow{\quad s \quad}$					

The degrees corresponding to rows and columns in the above diagrams are respectively the form degree and the ghost number. Notice that the form

$$c + A + B^* + \psi^* + \phi^*$$

is formally a "connection" with "curvature"

$$\phi + \psi - B + A^* + c^*.$$

So the field B^* allows the replacement, in the BV complex, of F_A with $-B$ in the structure equations. This is consistent with transformation (IX.7) and with integration by parts in the functional integral.

At the critical point $B=-F_A$, or equivalently, if we set $B^*=0$, we obtain, from the previous complex, the equations (II.9) and (II.10).

In conclusion topological field theories of the cohomological type are described by the structure equations and the Bianchi identities of a suitable bundle whose cohomology is some kind of equivariant cohomology. In topological field theories which are not of cohomological type, the above description is not possible, but the BV mechanism, with the introduction of fields with negative ghost number, allows us to recover, formally *off-shell*, the algebraic structure of equivariant cohomology.

X. Gauge fixing and orbits

In this section we would like to report briefly on the gauge-fixing problem for the action functional (I.4).

The results of sections **VI** and **VII** imply that we have an exact BRST symmetry when we consider the space of fields $K^*(\mathrm{T}\mathcal{A}\times\mathrm{T}\mathcal{B})$ and the symmetry group $\mathrm{T}\mathcal{G}$.

The action (I.4) is invariant under $\mathcal{G}_{\mathrm{aff}}$ and hence under its subgroup $\mathrm{T}\mathcal{G}$.

So the action (I.4) is naturally defined on the space

$$\frac{K^*(\mathrm{T}\mathcal{A}\times\mathrm{T}\mathcal{B})}{\mathrm{T}\mathcal{G}}=K^*\left(\frac{\mathrm{T}\mathcal{A}\times\mathrm{T}\mathcal{B}}{\mathrm{T}\mathcal{G}}\right);$$

the last identity is a consequence of the fact that the curvature map K (VII.7) commutes with the action of $\mathcal{G}$ and hence descends to a map

$$K\colon\frac{\mathcal{A}}{\mathcal{G}}\to\frac{\mathcal{A}\times\mathcal{B}}{\mathcal{G}}.$$

We notice that the following bundles are isomorphic:

$$\frac{K^*(\mathrm{T}\mathcal{A}\times\mathrm{T}\mathcal{B})}{\mathrm{T}\mathcal{G}}\simeq\frac{K^*(\mathrm{H}\mathcal{A}\times\mathrm{T}\mathcal{B})}{\mathcal{G}},\tag{X.1}$$

the isomorphism being induced by the linear map

$$[A,\eta,B]_{\mathrm{T}\mathcal{G}}\rightsquigarrow[A,\eta^{\mathrm{H}},B-d_A\eta^{\mathrm{V}}]_{\mathcal{G}}.\tag{X.2}$$

Here $\mathrm{H}\mathcal{A}$ denotes the horizontal bundle defined by a connection c on $\mathcal{A}$ and the superscript V and H denote respectively the vertical and the horizontal projections.

In order to show that this map is well-defined on the equivalence classes we compute

$$\left(A^g,Ad_{g^{-1}}\eta+d_A\xi,Ad_{g^{-1}}B+[F_{A^g},\xi]\right)\rightsquigarrow$$

$$\left(A^g, (Ad_{g^{-1}}\eta)^{\mathrm{H}}, Ad_{g^{-1}}B + [F_{A^g}, \xi] - d_{A^g}\left((Ad_{g^{-1}}\eta)^{\mathrm{V}} + d_{A^g}\xi\right) = \right.$$
$$\left(A^g, (Ad_{g^{-1}}\eta)^{\mathrm{H}}, Ad_{g^{-1}}B - Ad_{g^{-1}}\left(d_A\eta^{\mathrm{V}}\right)\right).$$

It is immediate to check that (X.2) is injective and surjective.

The action functional (I.4) is also invariant under the subgroup of $\mathcal{G}_{\mathrm{aff}}$ given by the translations by $\Omega^1(M, \operatorname{ad} P)$

$$(A, \eta, B) \rightsquigarrow (A, \eta + \tau, B + d_A\tau) \quad \tau \in \Omega^1(M, \operatorname{ad} P)\,. \tag{X.3}$$

When our space of fields is represented by $\dfrac{K^*(\mathrm{H}\mathcal{A} \times \mathrm{T}\mathcal{B})}{\mathcal{G}}$, then we have to consider triples (A, η, B) with $\eta \in \mathrm{H}_A\mathcal{A}$. The action (X.3) now becomes:

$$(A, \eta, B) \rightsquigarrow (A, \eta + \tau^H, B + d_A\tau^H) \quad \tau \in \Omega^1(M, \operatorname{ad} P)\,. \tag{X.4}$$

The above action is of course not free, but the space of orbits is still a manifold, namely it is

$$\frac{\mathcal{A} \times \mathcal{B}}{\mathcal{G}} \sim \frac{K_2^*\mathrm{T}\mathcal{B}}{\mathcal{G}}. \tag{X.5}$$

Different gauges are possible to describe the above orbit spaces. Let us clarify first what we mean by “choice of the gauge” in the present framework.

We have the vector bundle

$$\mathcal{V} \triangleq K^*(\mathrm{H}\mathcal{A} \times \mathrm{T}\mathcal{B}) \to \mathcal{A} \tag{X.6}$$

which is isomorphic to the vector bundle

$$(\mathrm{H}\mathcal{A} \times \mathcal{B}) \to \mathcal{A}.$$

The $\mathcal{G}$-invariant fiber metric in $\mathrm{H}\mathcal{A}$ and in $\mathcal{B}$ induces a $\mathcal{G}$-invariant fiber metric in $\mathcal{V}$.

Fixing a gauge means finding a vector bundle $\mathcal{W}$ so that we have a $\mathcal{G}$-equivariant splitting

$$\mathcal{V} = \mathrm{H}\mathcal{A} \oplus \mathcal{W}.$$

The first obvious choice is to choose

$$\mathcal{W} = \mathcal{A} \times \mathcal{B}. \tag{X.7}$$

In other words the gauge condition reads: $\eta = 0$, meaning that the elements of $\mathcal{W}$ have zero projection on $\mathrm{H}\mathcal{A}$. We refer to the above gauge as to the *trivial gauge*.

In order to discuss the second choice of the gauge, we restrict A to belong to the open subset of $\mathcal{A}$, where the dimension of the space $\mathrm{Harm}^1_A(M, \operatorname{ad} P)$ of harmonic 1-forms has the lowest dimension (possibly zero). Let us denote this open set by the symbol $\hat{\mathcal{A}}$.

The symmetry (X.4) indicates that we have at our disposal a second gauge fixing: namely we can impose on B the following condition $\langle B, d_A\eta\rangle = 0$, i.e., $\langle d_A^*B, \eta\rangle = 0$, $\forall \eta \in \mathrm{H}\mathcal{A}$. Here $\langle \bullet, \bullet\rangle$ denotes the inner product in $\Omega^*(M, \operatorname{ad} P)$.

Our second gauge fixing is then a condition on both η and $B \in \mathcal{B}$; namely, we require $\eta \in \mathrm{H}_A\hat{\mathcal{A}}$ and $B \in \hat{\mathcal{B}}$, where $\hat{\mathcal{B}}$ is a vector bundle of 2-forms over $\hat{\mathcal{A}}$ defined by the condition

$$\hat{\mathcal{B}}_A \triangleq \left\{ B \in \mathcal{B} \middle| d_A^* B \in \mathrm{Im}\left(d_A\big|_{\Omega^0(M,\mathrm{ad}\,P)} \right) \right\}. \tag{X.8}$$

This choice corresponds to defining

$$\mathcal{W}_A = \left(\mathrm{H}_A\hat{\mathcal{A}} \ominus \mathrm{Harm}^1_A(M, \mathrm{ad}\,P) \right) \oplus \hat{\mathcal{B}}_A.$$

Notice that the orthogonal projection of $\mathcal{W}_A$ on $\mathrm{H}_A\hat{\mathcal{A}}$ is not any more zero, but it is given by the orthogonal complement of the harmonic 1-forms.

We refer to the above gauge as the *covariant gauge.*

Finally we consider the *self-dual gauge.*

We start by considering the projection on self-dual 2-forms [11] : $P_+ : \Omega^2(M, \mathrm{ad}\,P) \to \Omega^2(M, \mathrm{ad}\,P)^+$. We now restrict ourselves to the open set of connections where the operator $D_A \triangleq \sqrt{2} P_+ d_A : \Omega^1(M, \mathrm{ad}\,P) \to \Omega^2(M, \mathrm{ad}\,P)^+$ is surjective. We denote by $\widetilde{\mathcal{A}}$ such an open set.

If we define also

$$D_A \triangleq \sqrt{2} d_A : \Omega^2(M, \mathrm{ad}\,P)^+ \to \Omega^3(M, \mathrm{ad}\,P)$$

and

$$D_A \triangleq d_A : \Omega^0(M, \mathrm{ad}\,P) \to \Omega^1(M, \mathrm{ad}\,P),$$

then we can consider the forms that are harmonic with respect to the operator D_A, for which we use the notation $\widetilde{\mathrm{Harm}}^*_A(M, \mathrm{ad}\,P)$.

The self-dual gauge condition is now defined as follows: $\eta \in \mathrm{H}_A\widetilde{\mathcal{A}}$ and $P_+ B = 0$.

In this way we have defined:

$$\mathcal{W}_A = \left(H_A\widetilde{\mathcal{A}} \ominus \widetilde{\mathrm{Harm}}^1_A(M, \mathrm{ad}\,P) \right) \oplus \mathcal{B}^-,$$

where $\mathcal{B}^-$ is the space of anti-self-dual elements of $\mathcal{B}$.

Again the orthogonal projection of $\mathcal{W}_A$ on $\mathrm{H}_A\hat{\mathcal{A}}$ is not zero, but it is given by the orthogonal complement of the 1-forms that are harmonic with respect to the operator D_A.

Finally, we consider the equations of motion relevant to the action functional (I.4) and the corresponding extrema.

Taking the directional derivatives at (A, η, B) in the directions (τ, μ, C) we get the equations

$$\begin{aligned} -t\langle [\tau, \eta], B - d_A\eta \rangle + \langle B, *d_A\tau \rangle &= 0 \\ \langle d_A\mu, B - d_A\eta \rangle &= 0 \\ t\langle C, B - d_A\eta \rangle + \langle C, *F \rangle &= 0 \end{aligned}$$

Noticing that

$$\langle [\tau, \eta], U \rangle = -\langle \tau, *[*U, \eta] \rangle$$

for a generic 2-form U, we get

$$\begin{aligned} t[*B - *d_A\eta, \eta] + d_A B &= 0 \\ d_A^*(B - d_A\eta) &= 0 \\ t(B - d_A\eta) + *F_A &= 0 \end{aligned}$$

Owing to the Bianchi identity, the second equation is redundant and the first equation is equivalent to the Yang–Mills equation $d_A^* F_A = 0$. In summary the equations of motions of (I.4) read

$$\begin{aligned} d_A^* F_A &= 0 \\ t(B - d_A\eta) + *F_A &= 0. \end{aligned} \qquad \text{(X.9)}$$

At $t = 1$ the action functional (I.4) reads

$$S_{BFYM,\eta} = \frac{1}{2}\langle B - d_A\eta + *F_A, B - d_A\eta + *F_A\rangle - YM(A). \qquad \text{(X.10)}$$

The points (A, η, B) satisfying the equation

$$B = d_A\eta - *F_A \qquad \text{(X.11)}$$

are the minima of (X.10) once we have fixed the connection A.

When the condition (X.11) is satisfied, then (X.10) reduces to the Yang–Mills action functional (up to a sign).

XI. Conclusions

In this paper we have first discussed the differential geometry of the tangent bundle $\mathrm{T}P$ of a G-principal bundle P. It is known that $\mathrm{T}P$ is a $\mathrm{T}G$-principal bundle.

We have shown that the group of gauge transformations of $\mathrm{T}P$ includes, as a proper subgroup, the tangent gauge group $\mathrm{T}\mathcal{G}$, namely, the tangent bundle of the group of gauge transformations on P.

The tangent gauge group acts on the tangent bundle of the space of connections $\mathcal{A}$ of P, and we have identified this tangent bundle with a subspace of the space of connections of $\mathrm{T}P$.

Then we have considered topological and non topological quantum field theories that admit the tangent gauge group as their symmetry group.

It is known that:

A) The structure equations and the Bianchi identities for the bundle

$$\frac{P \times \mathcal{A}}{\mathcal{G}}$$

can be interpreted as the BRST equations for a topological quantum field theory (see [3]).

B) By taking the restriction along a $\mathcal{G}$-fiber of the bundle $P \times \mathcal{A}$, one obtains from the above structure equations the BRST equations for the Yang–Mills theory.

In this paper we have first applied the “tangent functor” to the above cases A and B. Moreover, a special mixing of these two cases has been considered by taking the restriction to the $\mathcal{G}$-orbit of the bundle $\mathrm{T}P \times \mathrm{T}\mathcal{A}$.

In this way we have defined a theory that is “topological” when we consider the fields that are represented by tangent vectors to $\mathcal{A}$ and coincides with the Yang–Mills theory when we consider the connection itself.

The remarkable fact is that, in the above “semi-topological” theory, one is able to extend the BRST operator to fields represented by 2-forms. In other words one is allowed to consider field theories that include not only tangent vectors to the space of connections, but also tangent vectors to the space of the curvatures of connections.

The most important non-trivial example of these field theories is the 4-dimensional BFYM theory, which, in a separate paper [1], has been shown to be equivalent to the ordinary Yang–Mills theory. The fact that 2-forms are among the fields of this theory can be exploited by considering vacuum expectation values of observables depending on iterated loops (loops of loops) or on imbedded surfaces in a 4-dimensional manifold [4].

The gauge fixing problem for the BFYM theory is discussed in the last section. In particular the self-dual gauge and the covariant gauge are compared.

It is rather straightforward to extend all the above calculations to iterated tangent spaces. Hence we have constructed a BRST complex that includes—besides the connection A—$2^{n+1} - 1$ fields which are 1-forms, 2^n fields which are 2-forms and the relevant ghost fields. We are investigating possible relations between this infinite BRST complex and SUSY models.

Finally, we have shown that (pure) *topological* field theories of the BF type do not have a BRST operator related to some Lie algebra cohomology. In other words the BRST operator considered in [3] does *not* extend to 2-forms. The way of dealing with topological BF theories, from a field-theoretical point of view, is to resort to the Batalin–Vilkovisky formalism.

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